Algorithms

Augmenting Data Structures: Interval Trees

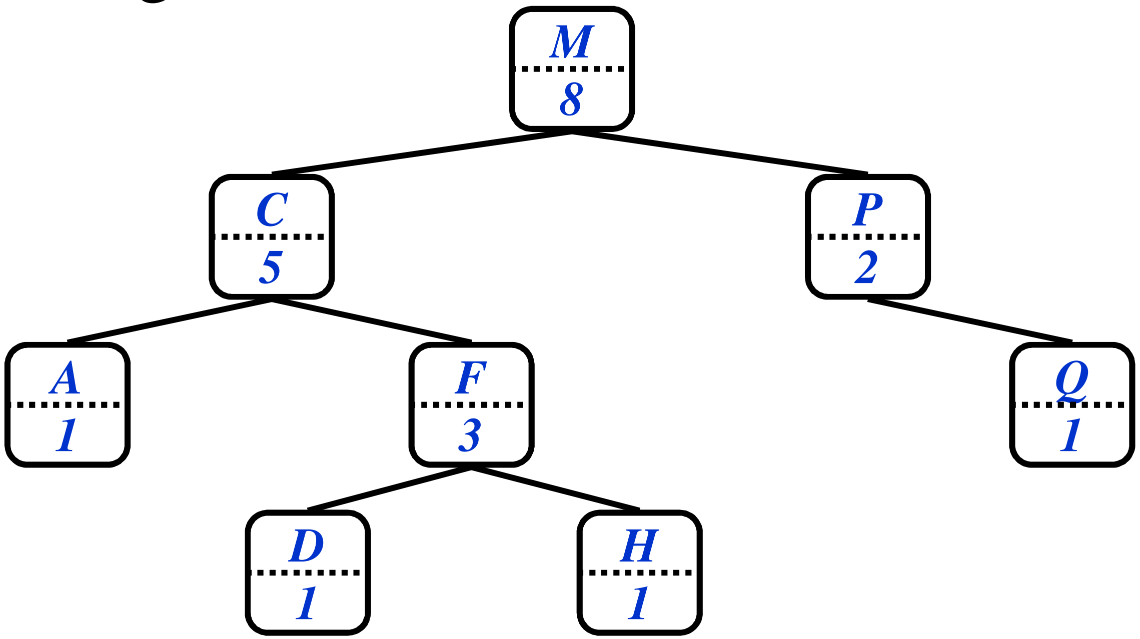
Review: Dynamic Order Statistics

- We've seen algorithms for finding the *i*th element of an unordered set in O(*n*) time
- *OS-Trees*: a structure to support finding the *i*th element of a dynamic set in O(lg *n*) time
  - Support standard dynamic set operations (**`Insert()`**, **`Delete()`**, **`Min()`**, **`Max()`**, **`Succ()`**, **`Pred()`**)
  - Also support these order statistic operations:

```
void OS-Select(root, i);
int OS-Rank(x);
```

Review: Order Statistic Trees

- OS Trees augment red-black trees:
  - Associate a *size* field with each node in the tree
  - **`x->size`** records the size of subtree rooted at **`x`**, including **`x`** itself:



Review: OS-Select

- Example: show OS-Select(*root*, 5):

```
OS-Select(x, i)
{
  r = x->left->size + 1;
  if (i == r)
    return x;
  else if (i < r)
    return OS-Select(x->left, i);
  else
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```

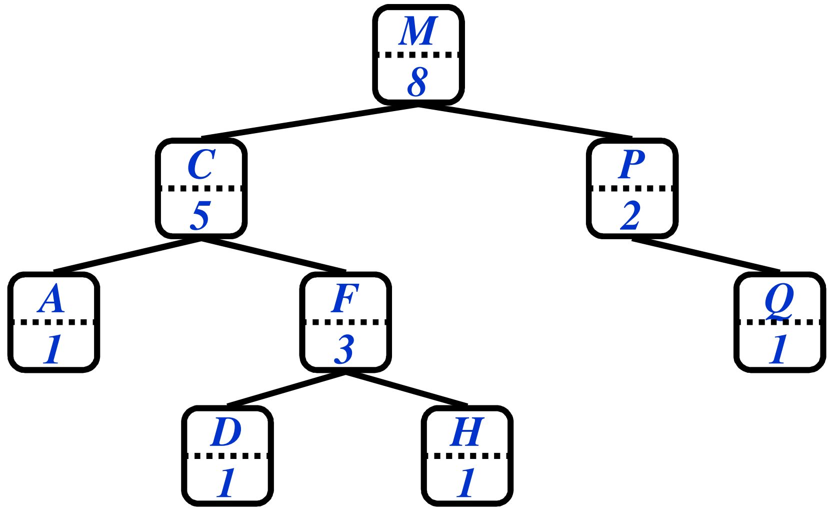

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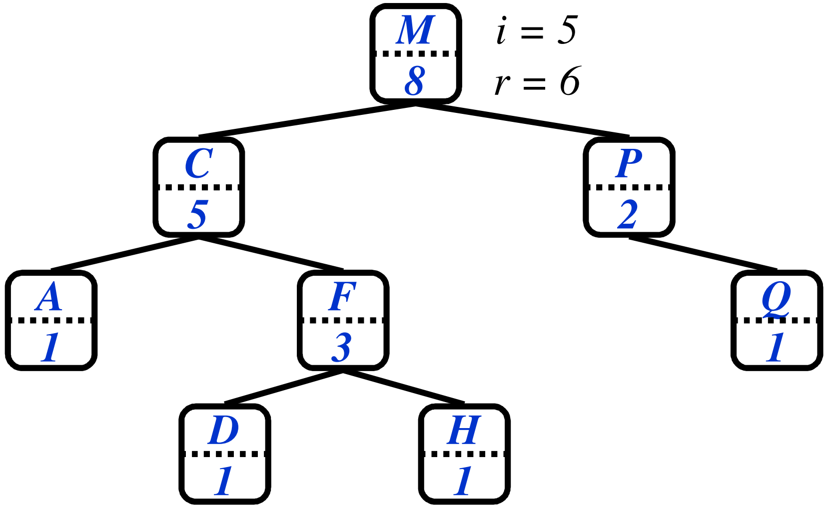

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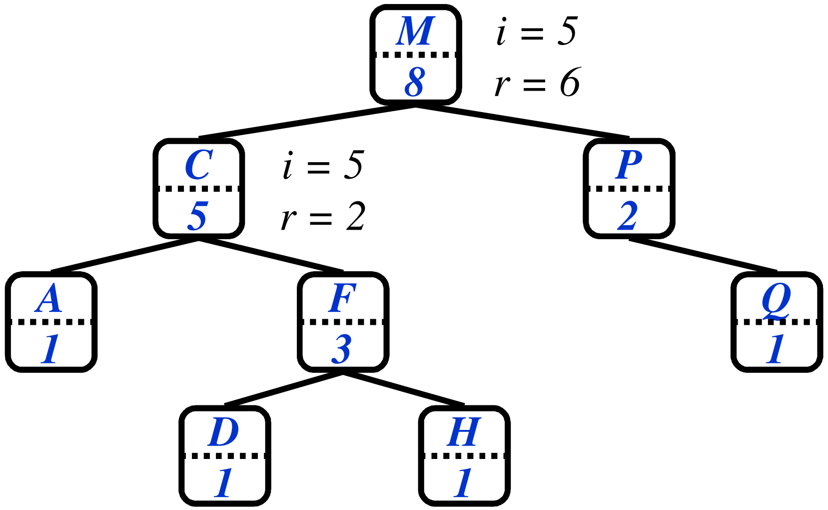

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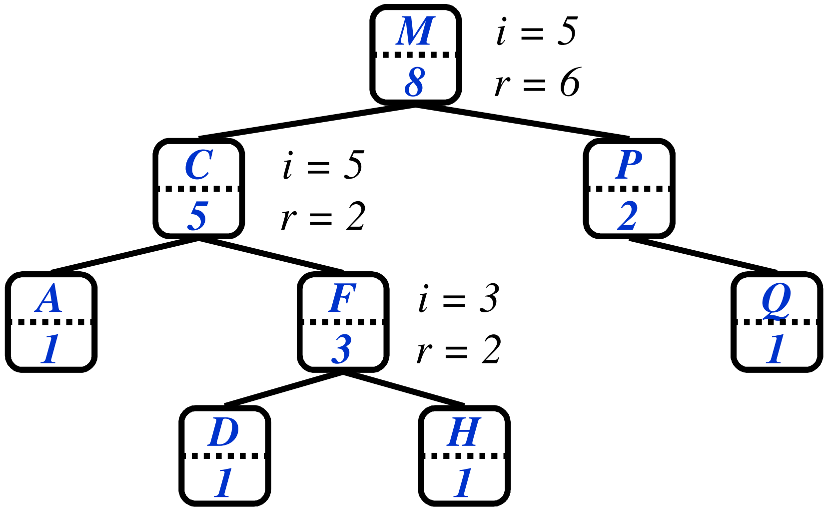

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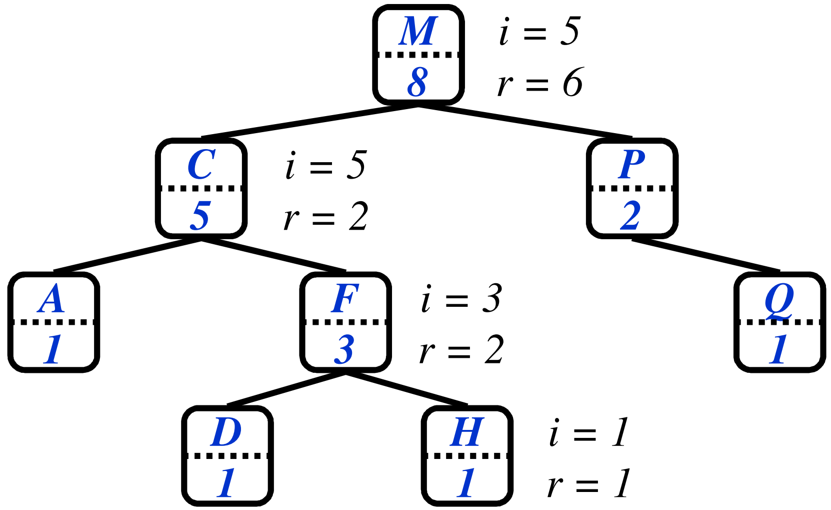

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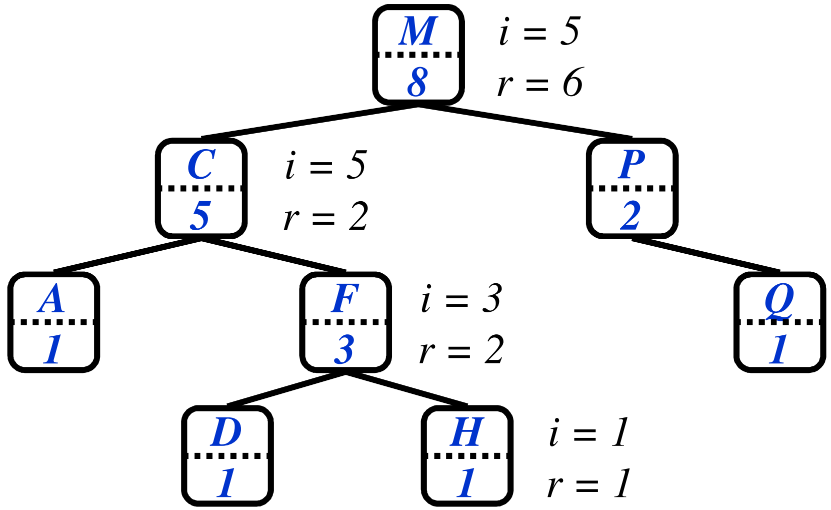
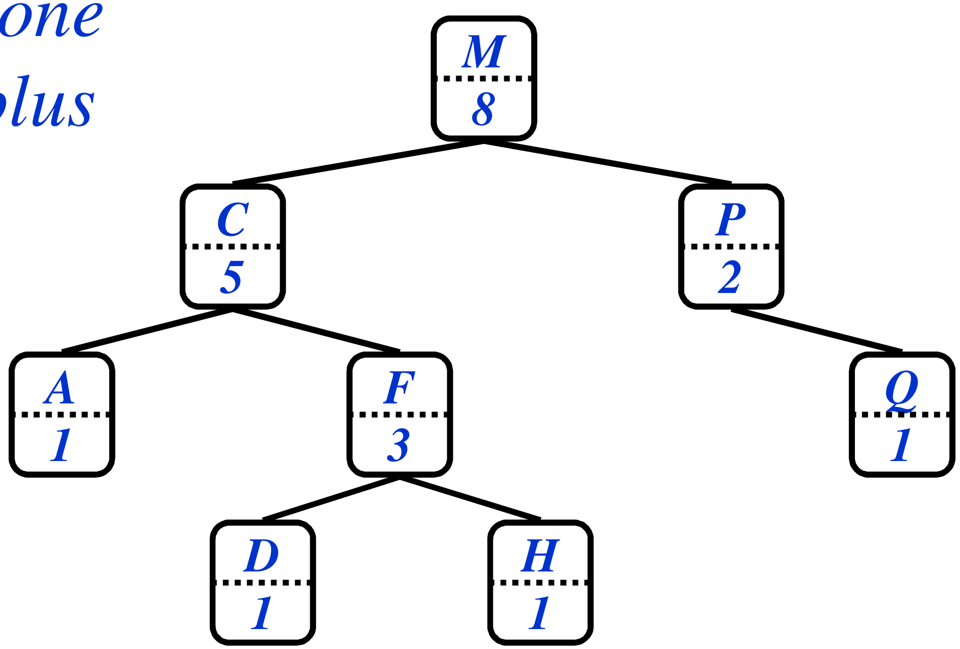


Note: use a sentinel NIL element at the leaves with size = 0 to simplify code, avoid testing for NULL

Review: Determining The Rank Of An Element

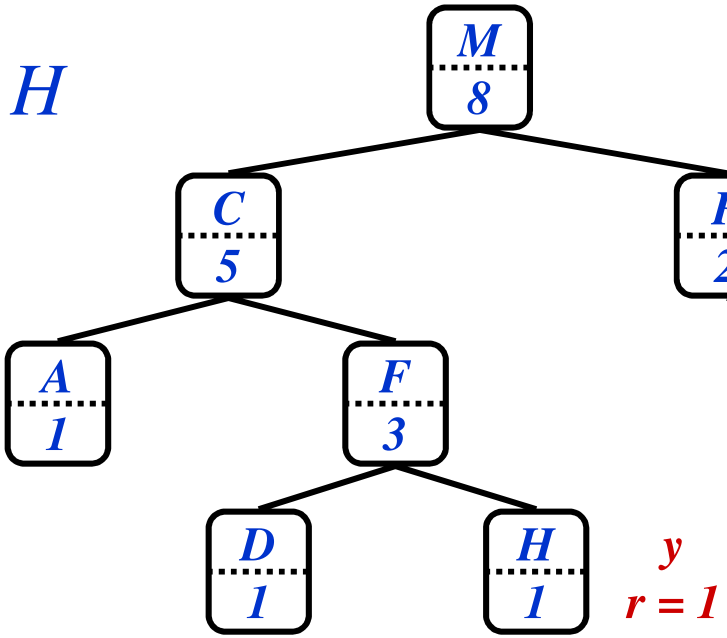
Idea: rank of right child x is one more than its parent's rank, plus the size of x's left subtree

```
OS-Rank(T, x)
{
    r = x->left->size + 1;
    y = x;
    while (y != T->root)
        if (y == y->p->right)
            r = r + y->p->left->size + 1;
        y = y->p;
    return r;
}
```



Review: Determining The Rank Of An Element

Example 1:
find rank of element with key H

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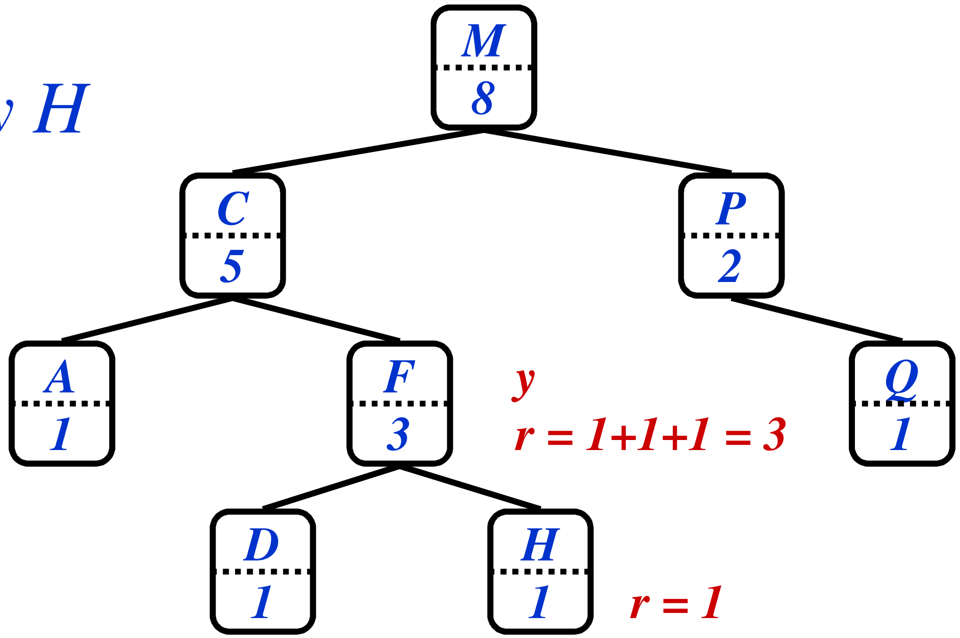


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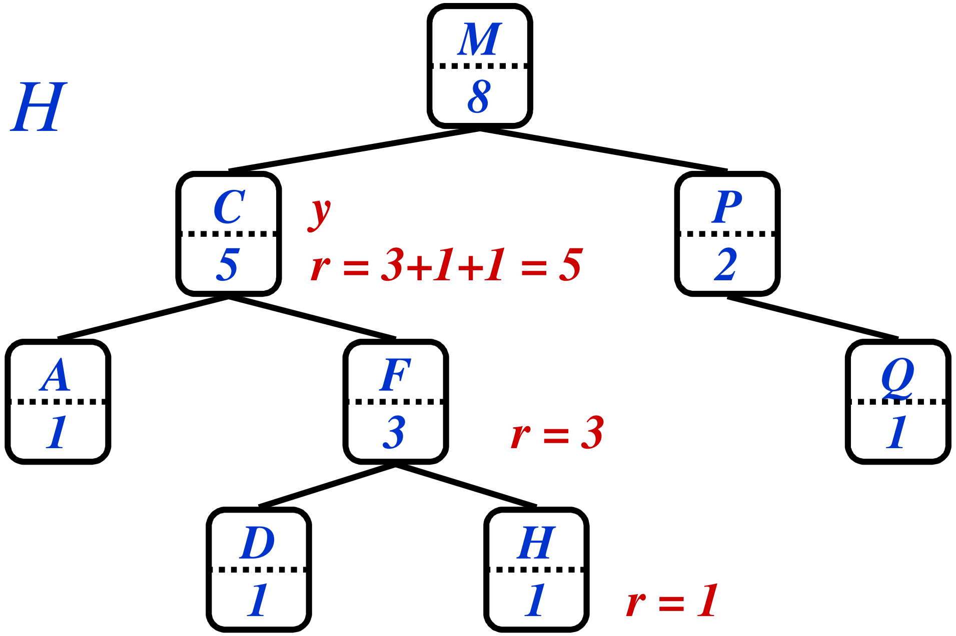


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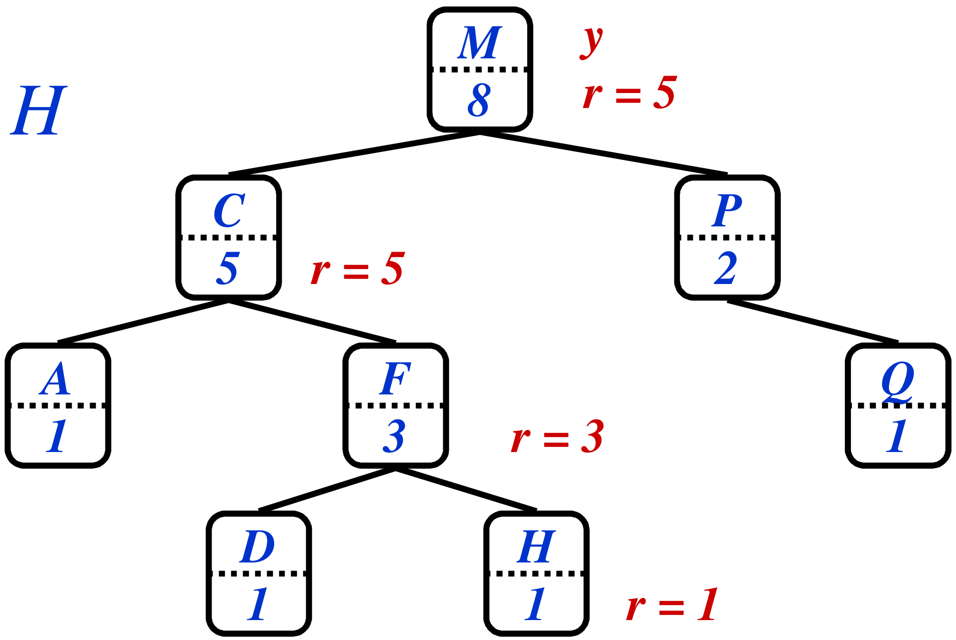


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```

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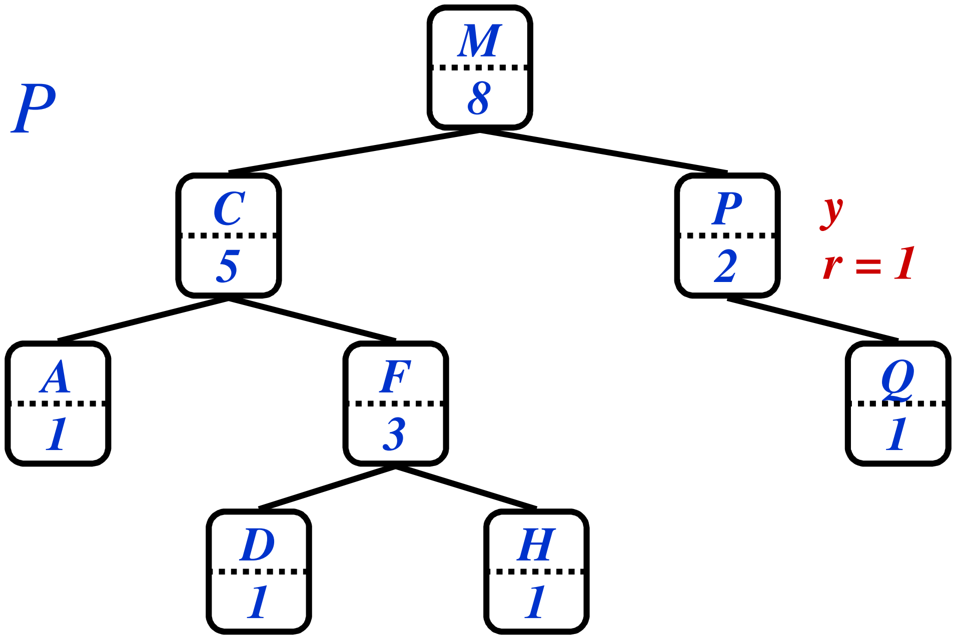
Example 2:

find rank of element with key P

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}
```

Review: Determining The Rank Of An Element

Example 2:

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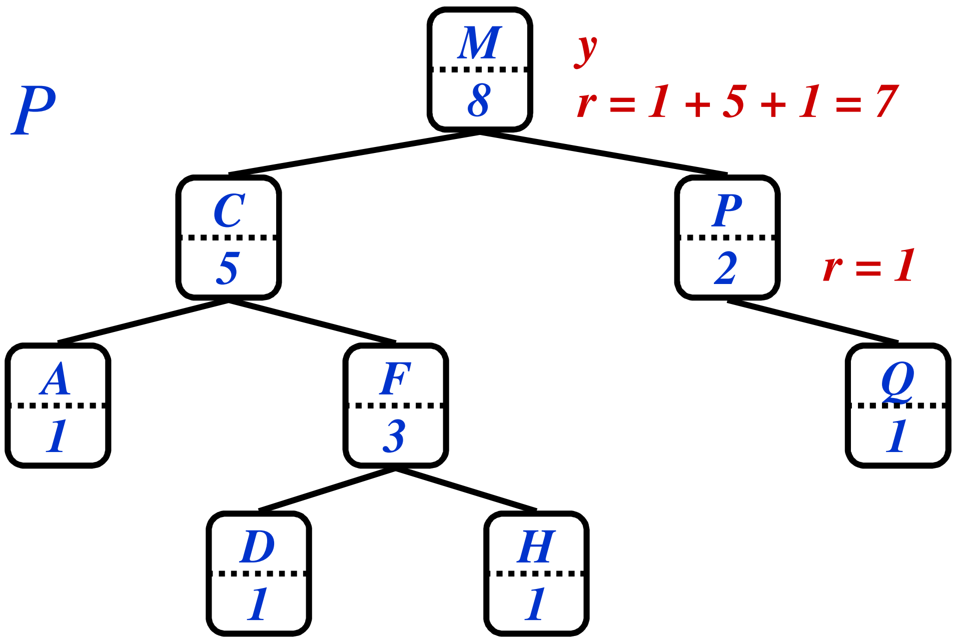


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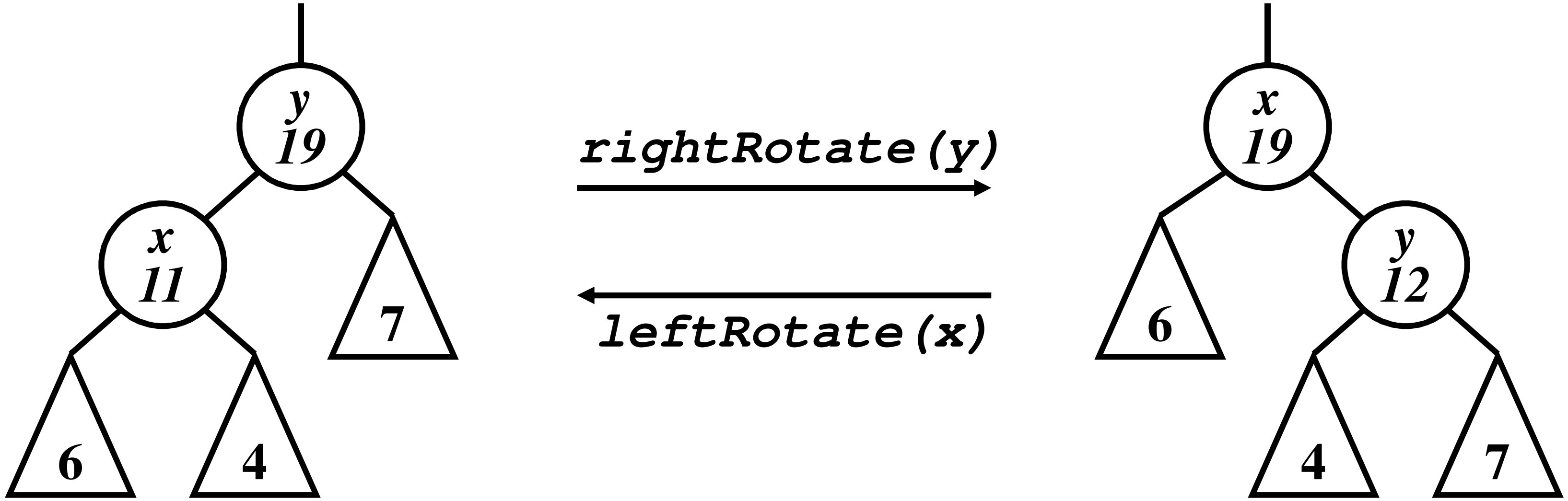
        y = y->p;
    return r;
}
```

Review: Maintaining Subtree Sizes

- So by keeping subtree sizes, order statistic operations can be done in O(lg n) time
- Next: maintain sizes during Insert() and Delete() operations
  - Insert(): Increment size fields of nodes traversed during search down the tree
  - Delete(): Decrement sizes along a path from the deleted node to the root
  - Both: Update sizes correctly during rotations

Reivew: Maintaining Subtree Sizes



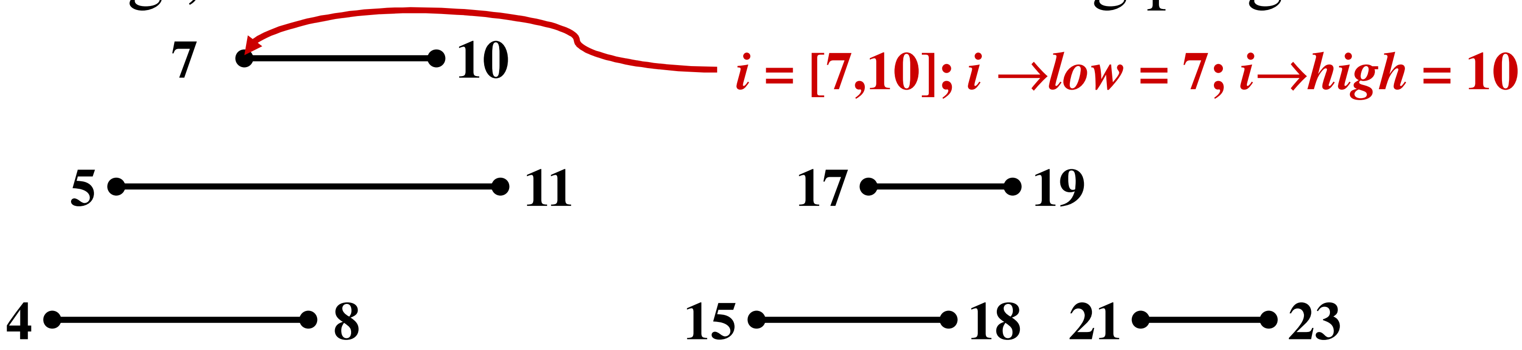
- Note that rotation invalidates only *x* and *y*
- Can recalculate their sizes in constant time
- Thm 15.1: can compute any property in O(lg n) time that depends only on node, left child, and right child

Review: Methodology For Augmenting Data Structures

- Choose underlying data structure
- Determine additional information to maintain
- Verify that information can be maintained for operations that modify the structure
- Develop new operations

Interval Trees

- The problem: maintain a set of intervals
  - E.g., time intervals for a scheduling program:



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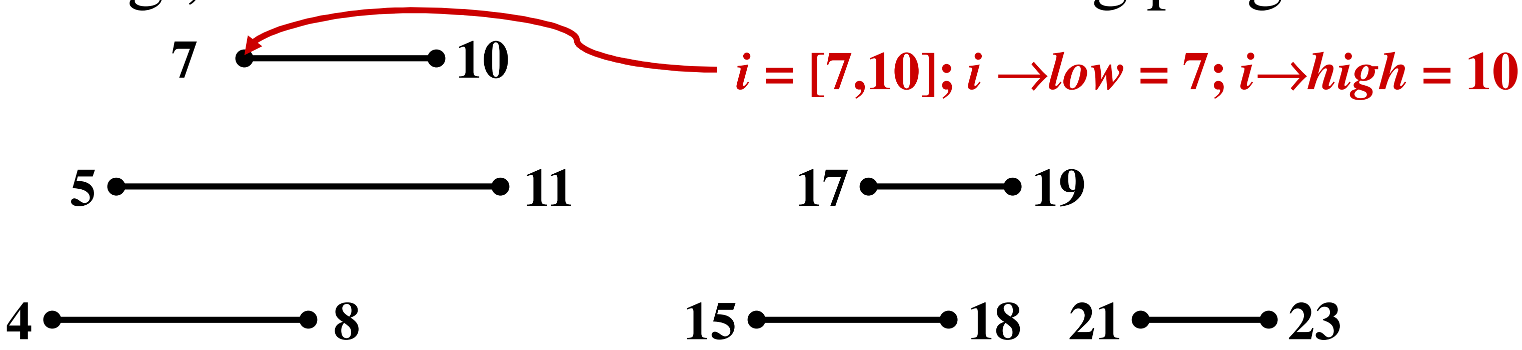


  - Query: find an interval in the set that overlaps a given query interval
    - [14,16] → [15,18]
    - [16,19] → [15,18] or [17,19]
    - [12,14] → NULL

Interval Trees

- Following the methodology:
  - Pick underlying data structure
  - Decide what additional information to store
  - Figure out how to maintain the information
  - Develop the desired new operations

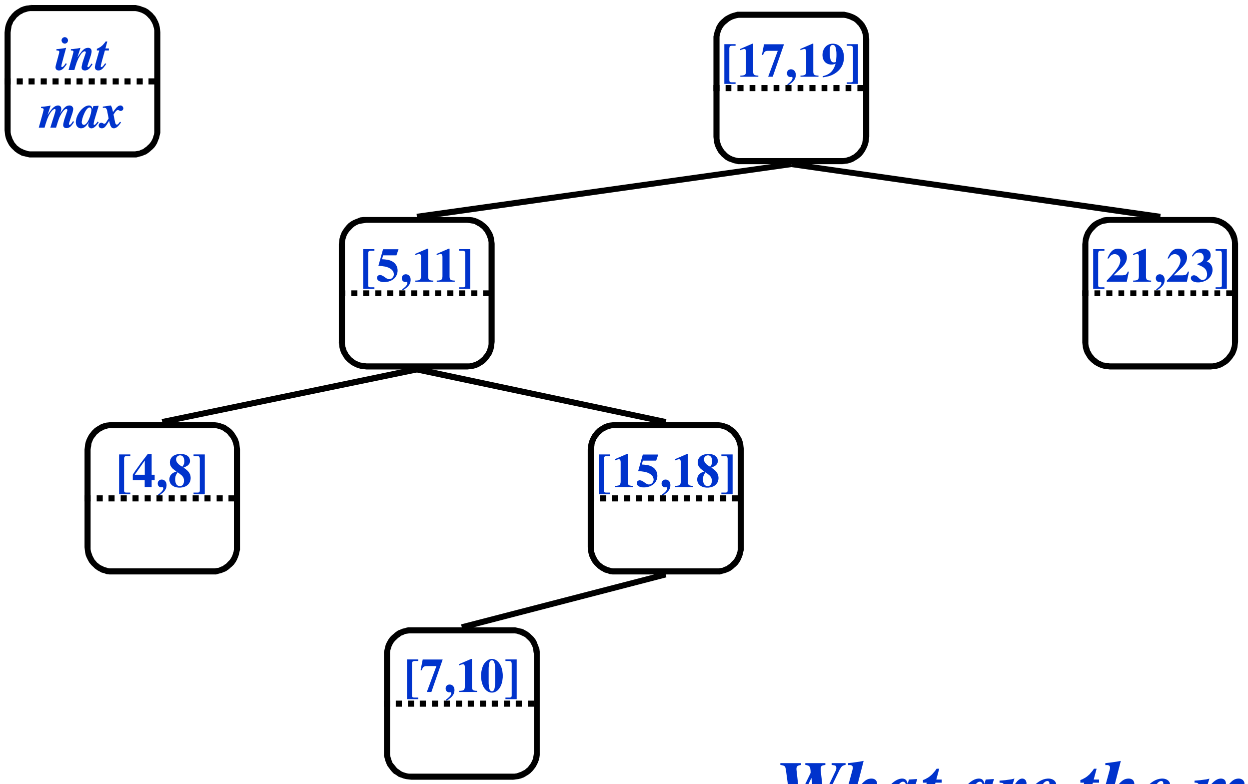
Interval Trees

- Following the methodology:
  - *Pick underlying data structure*
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - Decide what additional information to store
  - Figure out how to maintain the information
  - Develop the desired new operations

Interval Trees

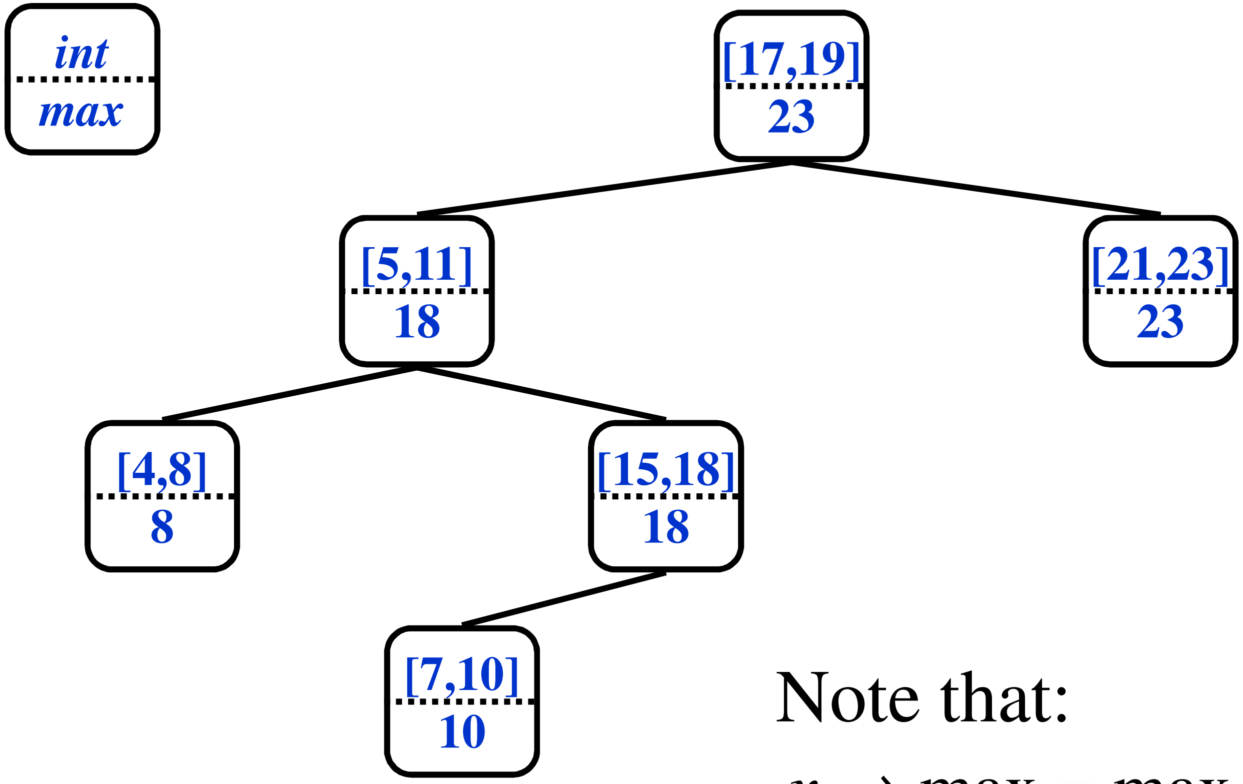
- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - *Decide what additional information to store*
    - We will store *max*, the maximum endpoint in the subtree rooted at *i*
  - Figure out how to maintain the information
  - Develop the desired new operations

Interval Trees



What are the max fields?

Interval Trees



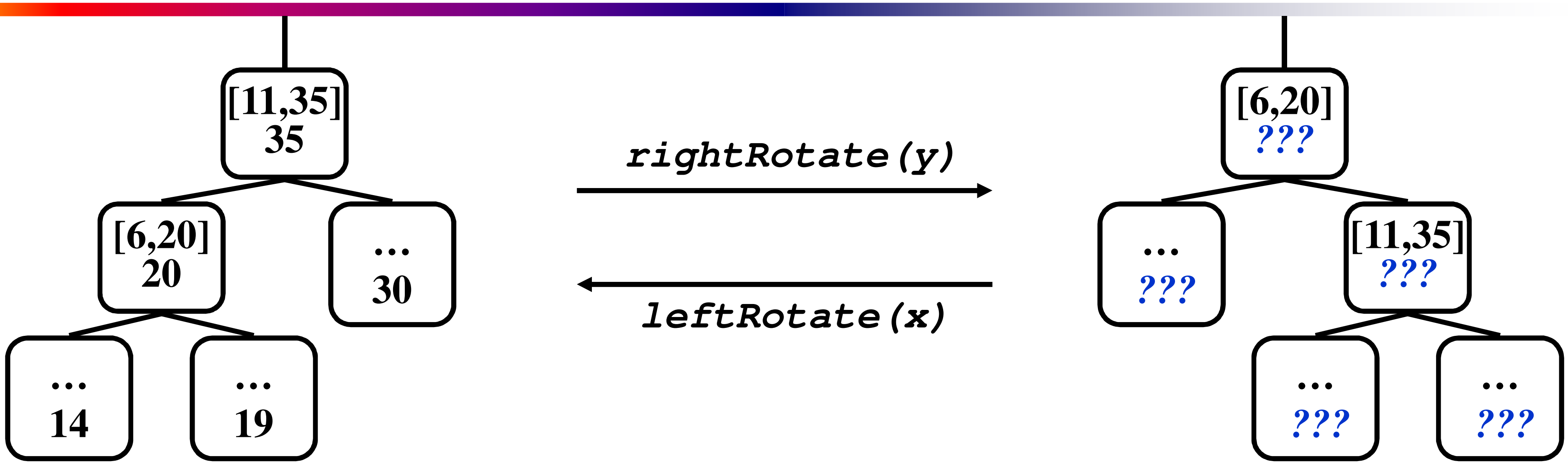
Note that:

$$x \to \max = \max \begin{cases} x \to high \\ x \to left \to \max \\ x \to right \to \max \end{cases}$$

Interval Trees

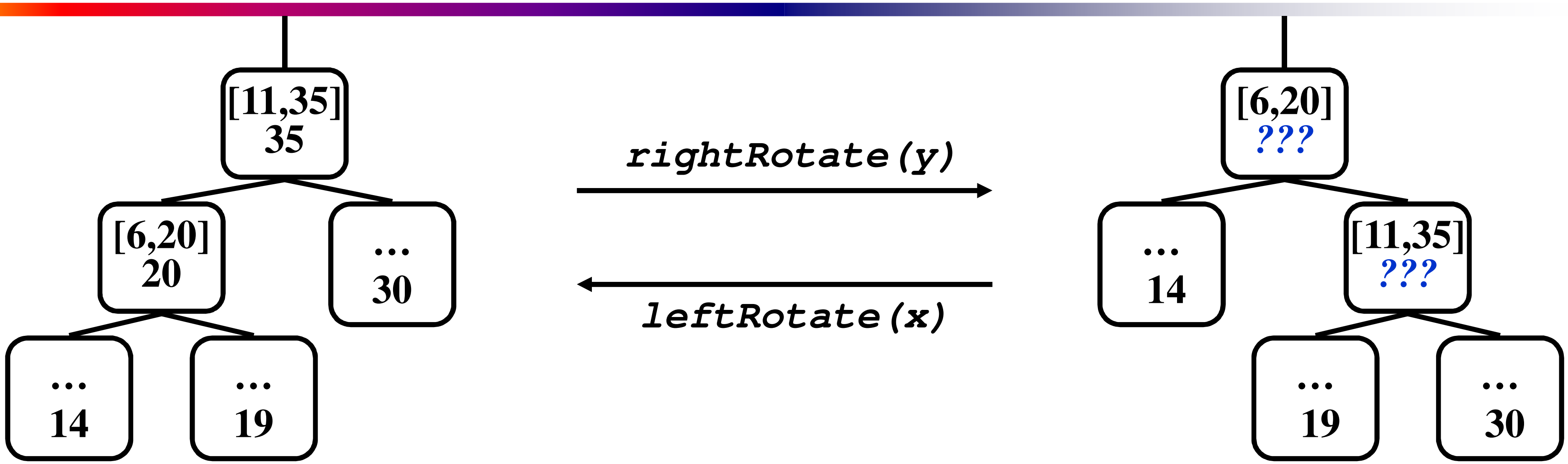
- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - Decide what additional information to store
    - Store the maximum endpoint in the subtree rooted at i
  - *Figure out how to maintain the information*
    - *How would we maintain max field for a BST?*
    - *What's different?*
  - Develop the desired new operations

Interval Trees



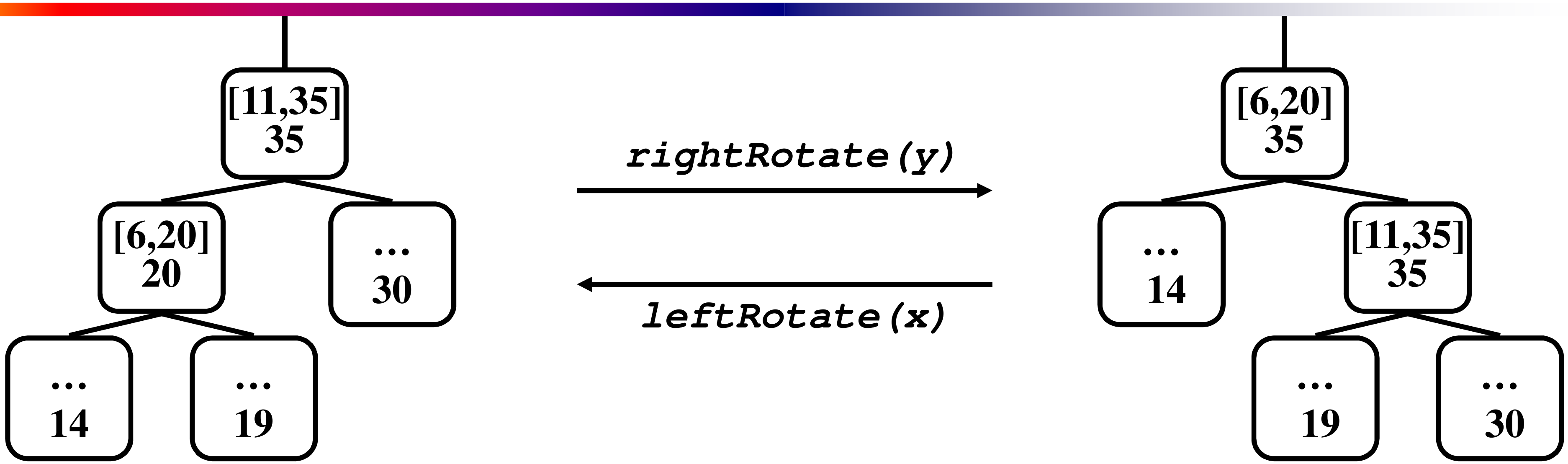
- *What are the new max values for the subtrees?*

Interval Trees



- *What are the new max values for the subtrees?*
- A: Unchanged
- *What are the new max values for x and y?*

Interval Trees



- *What are the new max values for the subtrees?*
- A: Unchanged
- *What are the new max values for x and y?*
- A: root value unchanged, recompute other

Interval Trees

- Following the methodology:
  - Pick underlying data structure
    - Red-black trees will store intervals, keyed on $i \rightarrow low$
  - Decide what additional information to store
    - Store the maximum endpoint in the subtree rooted at i
  - Figure out how to maintain the information
    - Insert: update max on way down, during rotations
    - Delete: similar
  - *Develop the desired new operations*

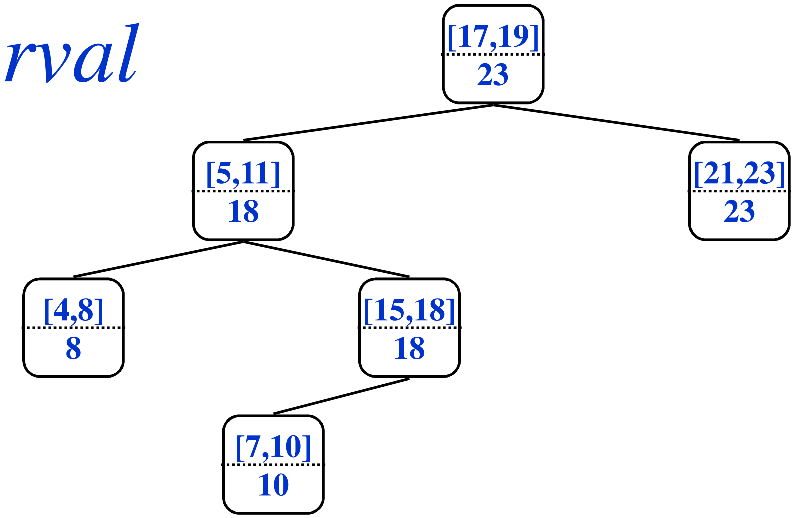
Searching Interval Trees

```
IntervalSearch(T, i)
{
    x = T->root;
    while (x != NULL && !overlap(i, x->interval))
        if (x->left != NULL && x->left->max ≥ i->low)
            x = x->left;
        else
            x = x->right;
    return x
}
```

- *What will be the running time?*

IntervalSearch() Example

- *Example: search for interval overlapping [14,16]*

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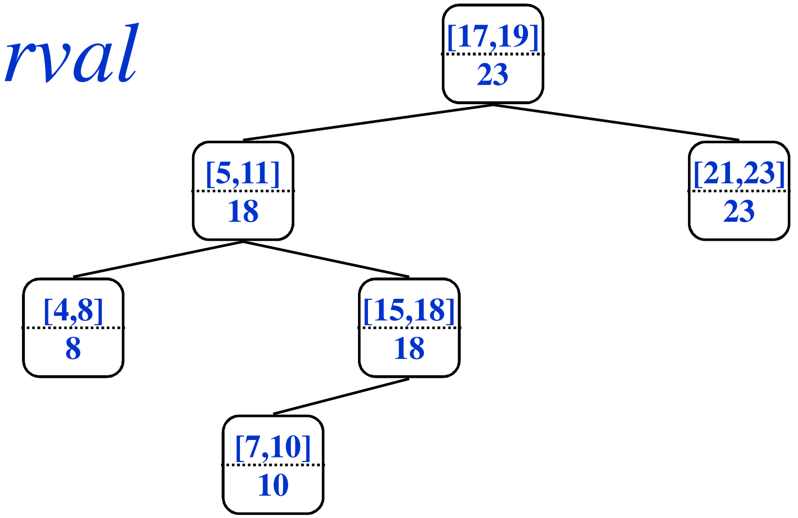
IntervalSearch() Example

- *Example: search for interval overlapping [12,14]*

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Correctness of IntervalSearch()

- Key idea: need to check only 1 of node’s 2 children
  - Case 1: search goes right
    - Show that ∃ overlap in right subtree, or no overlap at all
  - Case 2: search goes left
    - Show that ∃ overlap in left subtree, or no overlap at all

Correctness of IntervalSearch()

- Case 1: if search goes right, ∃ overlap in the right subtree or no overlap in either subtree
  - If ∃ overlap in right subtree, we're done
  - Otherwise:
    - $x \rightarrow left = NULL$, or $x \rightarrow left \rightarrow max < x \rightarrow low$ (*Why?*)
    - Thus, no overlap in left subtree!

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while (x != NULL && !overlap(i, x->interval))
        if (x->left != NULL && x->left->max ≥ i->low)
            x = x->left;
        else
            x = x->right;
    return x;
```

Correctness of IntervalSearch()

- Case 2: if search goes left, ∃ overlap in the left subtree or no overlap in either subtree
  - If ∃ overlap in left subtree, we're done
  - Otherwise:
    - i →low ≤ x →left →max, by branch condition
    - x →left →max = y →high for some y in left subtree
    - Since i and y don't overlap and i →low ≤ y →high, i →high < y →low
    - Since tree is sorted by low's, i →high < any low in right subtree
    - Thus, no overlap in right subtree

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while (x != NULL && !overlap(i, x->interval))
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