Rank And Nullity

Row Space and Column Space

If A is an $m \times n$ matrix, the subspace of $\mathbb{R}^n$ spanned by the row vectors if A is called the **row space** of A.

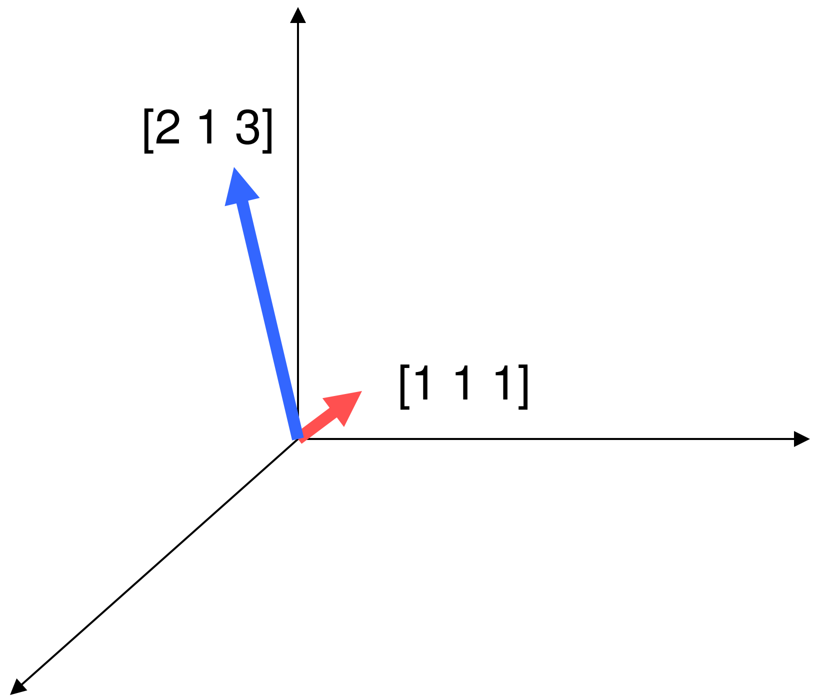
The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A.

Example

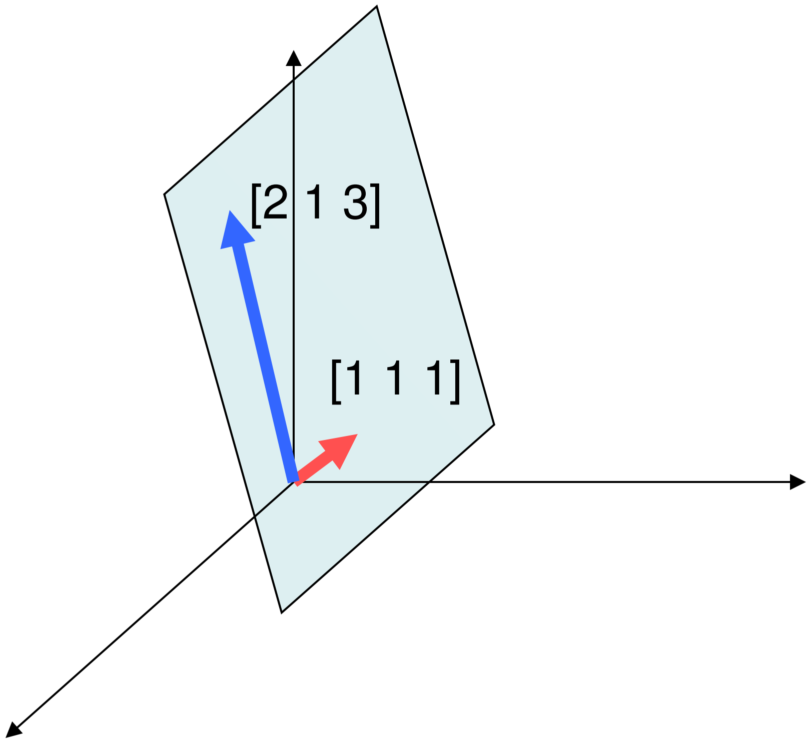
Find the row space and column space of

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

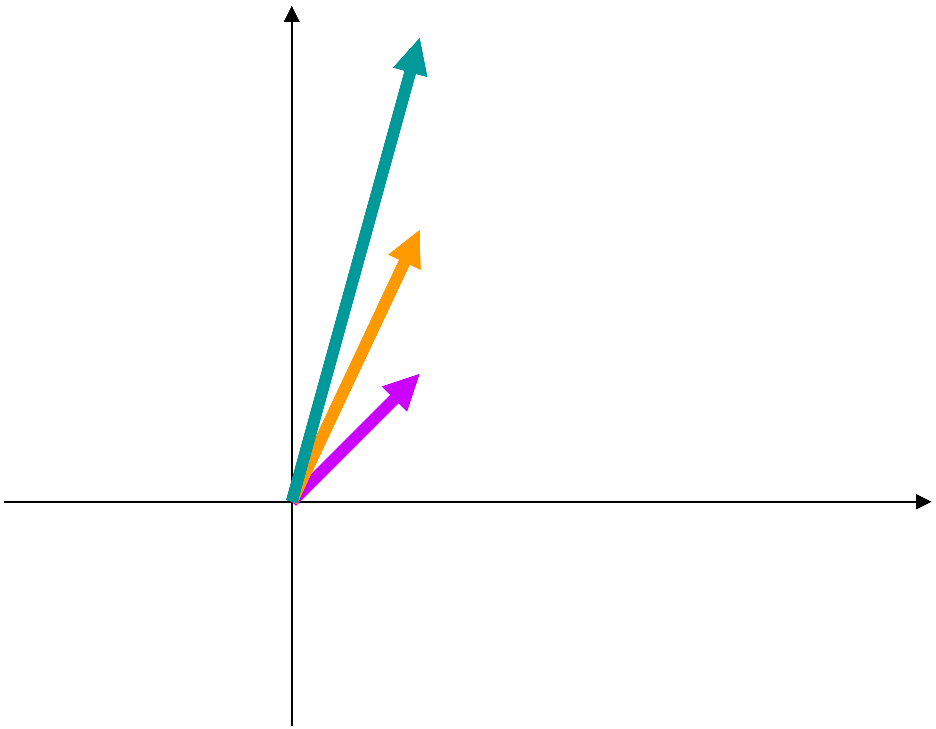
Row Space



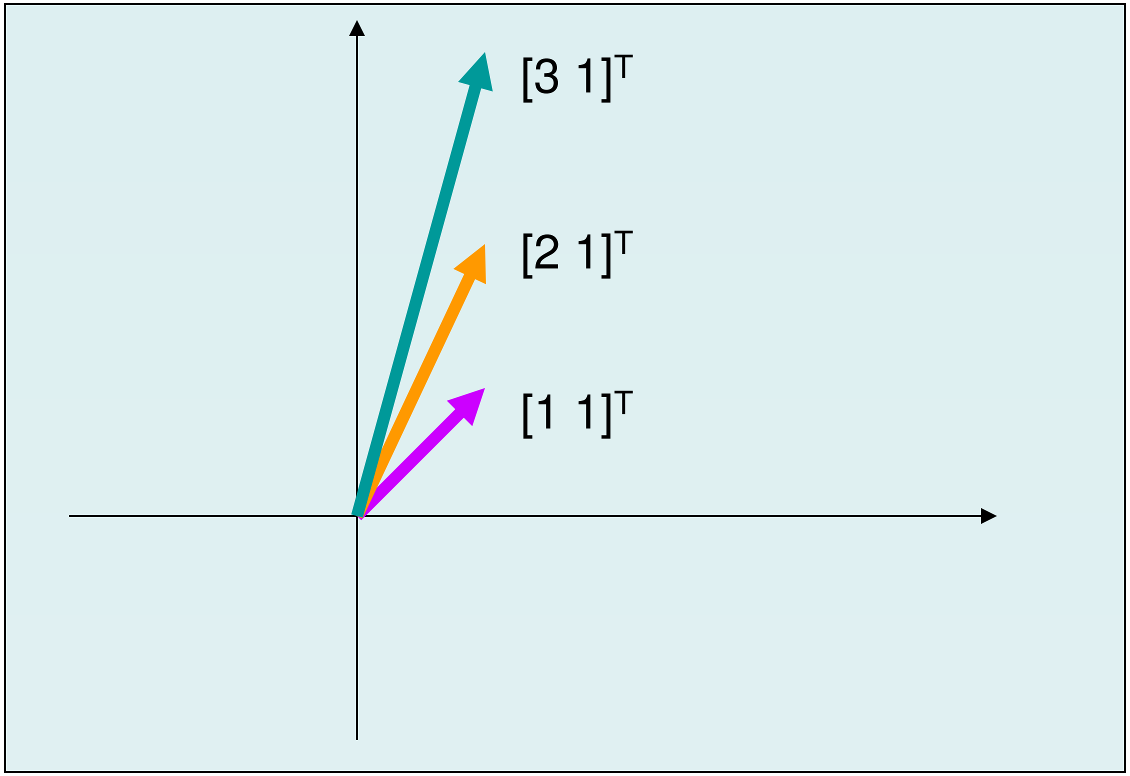
Row Space



Column Space



Column Space



Null Space

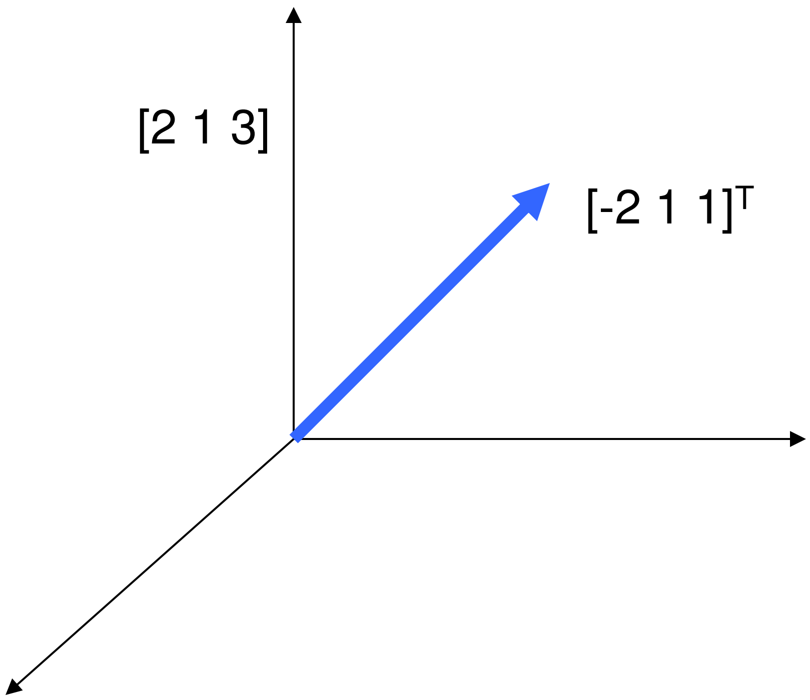
Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then N is a subspace of $\mathbb{R}^n$. N is called the **null space**.

Example

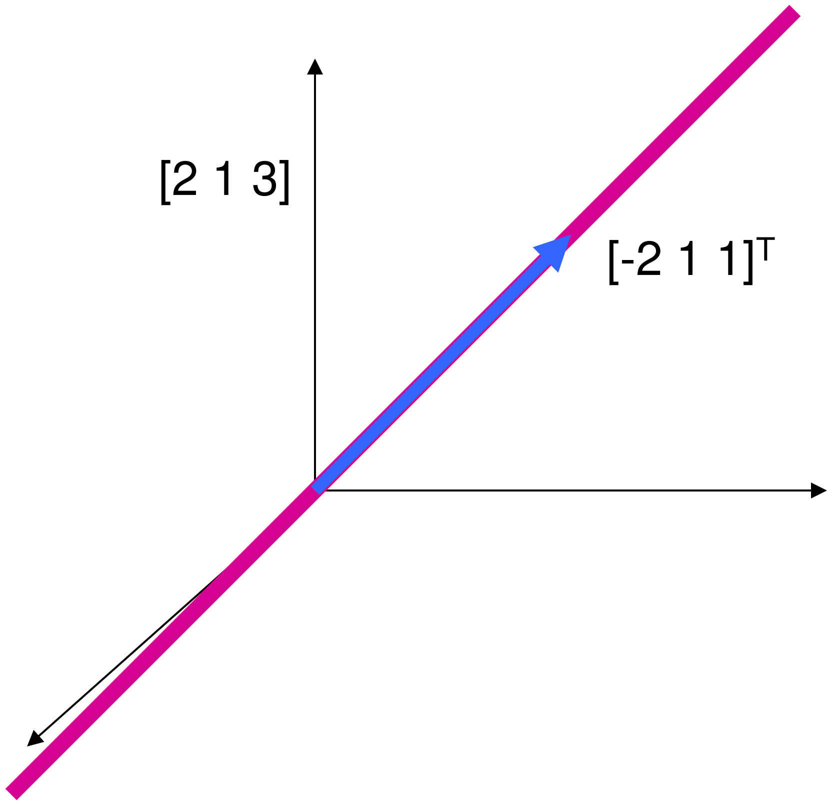
Find the null space of

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Row Space



Row Space



Effects of Row Operations

- Row operations do not alter row space.
- Row operations do not alter null space.
- Row operations do alter columns space.

  However, row operations do not change the linear relations of the columns. Thus, if A is a matrix with columns C_i and there are scalars α_i so that

  $$\alpha_1 C_1 + \alpha_2 C_2 + \ldots + \alpha_n C_n,$$

  then the same relation holds for any matrix B that is row equivalent to A.

Theorem

If A and B are row equivalent matrices, then

(1) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of A are linearly independent.

(2) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

Theorem

If a matrix R is in row-echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

Example

Find a basis and the dimension of the row space, the column space, and the null space for the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & -4 & -1 \\ 4 & -5 & 2 & 1 & 3 \\ 2 & -2 & 1 & 3 & 5 \end{bmatrix}$$

Result

If A is an $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.

Rank

The **rank** of a matrix A is the common dimension of the row and column spaces.

Nullity

The dimension of the nullspace of a matrix is called the **nullity** of the matrix.

The Rank-Nullity Theorem

If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity(A)} = n$$