Stokes' Theorem

If $\underset{\sim}{F}$ is a vector field on an open surface S and boundary of surface S is a closed curve c, therefore

$$\int_S curl\ \underset{\sim}{F} \cdot d\,\underset{\sim}{S} = \oint_c \underset{\sim}{F} \cdot d\,\underset{\sim}{r}$$

$$curl\ \underset{\sim}{F} = \nabla \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

Example:

Surface *S* is the combination of

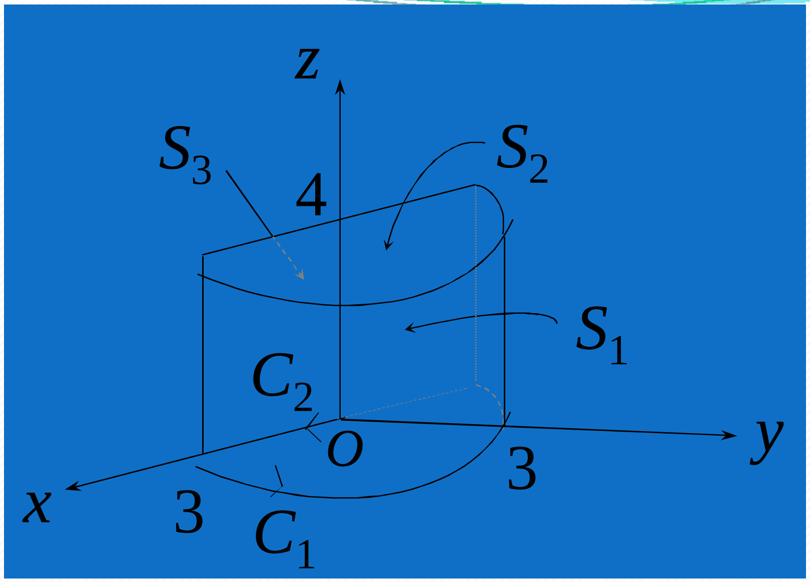
i) a part of the cylinder $x^2 + y^2 = 9$ $between$ $z = 0$ and $z = 4$ for $y \geq 0$.

ii) a half of the circle with radius 3 at $z = 4$, and

iii) $plane$ $y = 0$

If $\underset{\sim}{F} = z\underset{\sim}{i} + xy\underset{\sim}{j} + xz\underset{\sim}{k}$, prove Stokes' Theorem for this case.

Solution



We can divide surface S as

$S_1 : x^2 + y^2 = 9 \;\; \textit{for } 0 \le z \le 4 \textit{ and } y \ge 0$

$S_2 : z = 4$, half of the circle with radius 3

$S_3 : y = 0$

We can also mark the pieces of curve *C* as

C_1 : Perimeter of a half circle with radius 3.

C_2 : Straight line from (-3,0,0) to (3,0,0).

Let say, we choose to evaluate $\int_S curl\, \underset{\sim}{F} \cdot d\, \underset{\sim}{S}$ first.

Given $\underset{\sim}{F} = z\, \underset{\sim}{i} + xy\, \underset{\sim}{j} + xz\, \underset{\sim}{k}$

So,

$$curl\ \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & xy & xz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(xy) \right) \underset{\sim}{i} + \left(\frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}(xz) \right) \underset{\sim}{j}$$

$$+ \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(z) \right) \underset{\sim}{k}$$

$$= (1 - z)\,\underset{\sim}{j} + y\,\underset{\sim}{k}$$

By integrating each part of the surface,

$$(i) \quad For\ surface\ S_1 : x^2 + y^2 = 9,$$

$$\nabla S_1 = 2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j}$$

$$and \quad \left|\nabla S_1\right| = \sqrt{(2x)^2 + (2y)^2}$$

$$= 2\sqrt{x^2 + y^2} = 6$$

Then ,

$$\underset{\sim}{n} = \frac{\nabla S_1}{|\nabla S_1|} = \frac{2x\underset{\sim}{i} + 2y\underset{\sim}{j}}{6} = \frac{1}{3}(x\underset{\sim}{i} + y\underset{\sim}{j})$$

and

$$curl\ \underset{\sim}{F} \cdot \underset{\sim}{n} = \left((1-z)\underset{\sim}{j} + y\underset{\sim}{k} \right) \cdot \left(\frac{1}{3}x\underset{\sim}{i} + \frac{1}{3}y\underset{\sim}{j} \right)$$

$$= \frac{1}{3}y(1-z).$$

By using polar coordinate of cylinder (because $S_1 : x^2 + y^2 = 9$ is a part of the cylinder),

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$dS = \rho\, d\phi\, dz$$

where

$$\rho = 3, \quad 0 \le \phi \le \pi \quad dan \quad 0 \le z \le 4.$$

Therefore,

$$curl\ \underset{\sim}{F} \cdot \underset{\sim}{n} = \frac{1}{3} y(1-z)$$

$$= \frac{1}{3}\left(\rho \sin\phi\right)\left(1-z\right)$$

$$= \sin\phi(1-z)\ ;\ because\ \rho = 3$$

Also, $dS = 3\, d\phi\ dz$

$$\Rightarrow \int_{S_1} curl\ \underset{\sim}{F} \cdot d\ \underset{\sim}{S} = \int_{S_1} curl\ \underset{\sim}{F} \cdot \underset{\sim}{n}\ dS$$

$$= 3\int_{z=0}^{4} \int_{\phi=0}^{\pi} \left(\sin\phi(1 - z)\right) d\phi\, dz$$

$$= 3\int_{0}^{4} (1 - z)\left[-\cos\phi\right]_0^{\pi} dz$$

$$= 3\int_{0}^{4} (1 - z)(1 - (-1))dz$$

$$\vdots$$

$$= -24$$

(ii) For surface $S_2 : z = 4$, normal vector unit to the surface is $\underset{\sim}{n} = \underset{\sim}{k}.$

By using polar coordinate of plane ,

$$y = r\sin\theta, \quad z = 4 \quad dan \quad dS = r\,dr\,d\theta$$

$$where \quad 0 \le \mathrm{r} \le 3 \quad \text{and} \quad 0 \le \theta \le \pi.$$

$$\Rightarrow \quad curl\ \underset{\sim}{F} \cdot \underset{\sim}{n} = \left((1-z)\ \underset{\sim}{j} + y\ \underset{\sim}{k} \right) \cdot \left(\underset{\sim}{k} \right)$$

$$= y = r \sin\theta$$

$$\therefore \quad \int_{S_2} curl\ \underset{\sim}{F} \cdot d\ \underset{\sim}{S} = \int_{S_2} curl\ \underset{\sim}{F} \cdot \underset{\sim}{n}\, dS$$

$$= \int_{r=0}^{3} \int_{\theta=0}^{\pi} (r \sin\theta)(r\, dr\, d\theta)$$

$$= \int_{r=0}^{3} \int_{\theta=0}^{\pi} r^2 \sin\theta\, d\theta\, dr$$

$$\vdots$$

$$= 18$$

(iii) For surface $S_3 : y = 0$, normal vector unit

to the surface is $\underset{\sim}{n} = -\underset{\sim}{j}$.

$dS = dxdz$

The integration limits : $-3 \le x \le 3 \quad and \quad 0 \le z \le 4$

So,

$$curl\ \underset{\sim}{F} \cdot \underset{\sim}{n} = ((1-z)\underset{\sim}{j} + y\underset{\sim}{k}) \cdot (-\underset{\sim}{j})$$

$$= z - 1$$

Then,

$$\int_{S_3} curl\, \underset{\sim}{F} \,.\, d\, \underset{\sim}{S} = \int_{S_3} curl\, \underset{\sim}{F} \,.\, \underset{\sim}{n}\, dS$$

$$= \int_{x=-3}^{3} \int_{z=0}^{4} (z-1)\, dz dx$$

$$= \vdots$$

$$= 24.$$

$$\therefore \quad \int_{S} curl\, \underset{\sim}{F} \,.\, d\, \underset{\sim}{S} = \int_{S_1} curl\, \underset{\sim}{F} \,.\, d\, \underset{\sim}{S} + \int_{S_2} curl\, \underset{\sim}{F} \,.\, d\, \underset{\sim}{S} + \int_{S_3} curl\, \underset{\sim}{F} \,.\, d\, \underset{\sim}{S}$$

$$= -24 + 18 + 24$$

$$= 18.$$

Now, we evaluate $\oint_C \underset{\sim}{F} \, . \, d \underset{\sim}{r}$ for each pieces of the curve C.

i) C_1 is a half of the circle.

Therefore, integration for C_1 will be more easier if we use polar coordinate for plane with radius $r = 3,$ that is

$$x = 3\cos\theta, \quad y = 3\sin\theta \quad \text{dan} \quad z = 0$$

where $0 \leq \theta \leq \pi.$

$$\Rightarrow \quad \underset{\sim}{F} = z\underset{\sim}{i} + xy\underset{\sim}{j} + xz\underset{\sim}{k}$$

$$= (3\cos\theta)(3\sin\theta)\underset{\sim}{j}$$

$$= 9\sin\theta\cos\theta\underset{\sim}{j}$$

and $\quad dr = dx\underset{\sim}{i} + dy\underset{\sim}{j} + dz\underset{\sim}{k}$

$$= -3\sin\theta d\theta\underset{\sim}{i} + 3\cos\theta\, d\theta\underset{\sim}{j}.$$

From here,

$$\underset{\sim}{F} . d\underset{\sim}{r} = 27 \sin\theta \cos^2\theta \, d\theta.$$

$$\Rightarrow \quad \int_{C_1} \underset{\sim}{F} . d\underset{\sim}{r} = \int_0^{\pi} 27 \sin\theta \cos^2\theta \, d\theta$$

$$= \left[-9\cos^3\theta \right]_0^{\pi}$$

$$= 18.$$

ii) Curve C_2 is a straight line defined as

$$x = t, \quad y = 0 \quad and \quad z = 0, \quad \text{where} \quad -3 \le t \le 3.$$

Therefore, $\underset{\sim}{F} = z\underset{\sim}{i} + xy\underset{\sim}{j} + xz\underset{\sim}{k}$

$$= \underset{\sim}{0}.$$

$$\Rightarrow \int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0.$$

$$\therefore \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \oint_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} + \oint_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= 18 + 0$$

$$= 18.$$

We already show that

$$\int_S curl\, \underset{\sim}{F} \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$\Rightarrow$ Stokes' Theorem has been proved.